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Shoot-1.1 Package - User Guide

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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

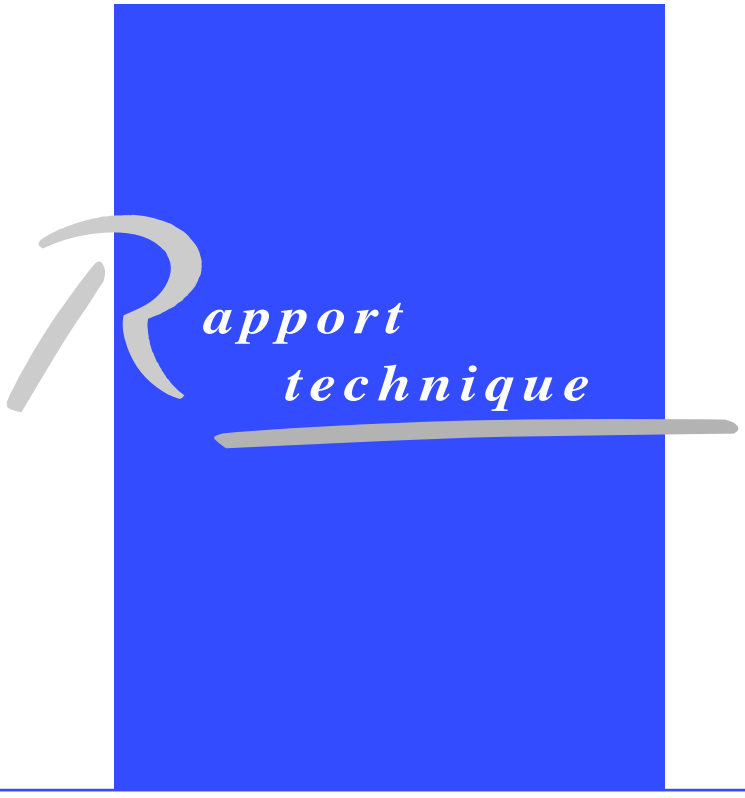
Shoot-1.1 Package - User Guide

Pierre Martinon — Joseph Gergaud

N° ????

Juin 2008

Thème NUM



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Shoot-1.1 Package - User Guide

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Abstract: This package implements a shooting method for solving boundary value problems, for instance resulting of the application of Pontryagin's Minimum Principle to an optimal control problem. The software is mostly Fortran90, with some third party Fortran77 codes for the numerical integration and nonlinear equations system. Its features include the handling of right hand side discontinuities (such as caused by a bang-bang control) for the integration of the trajectory and the computation of Jacobians for the shooting method. The particular case of singular arcs for optimal control problems is also addressed.

Key-words: Boundary value problem, discontinuous right hand side, optimal control, shooting method, singular arcs

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Package Shoot-1.1 - Guide Utilisateur

Résumé : Ce paquetage implémente une méthode de tir pour la résolution de problèmes aux deux bouts, tels que ceux obtenus par application du Principe du Minimum de Pontryaguine à un problème de contrôle optimal. Le logiciel est principalement écrit en Fortran90, et utilise des codes externes en Fortran77 pour l'intégration numérique et la résolution de systèmes non linéaires.Il prend notamment en compte les discontinuités du second membre, du point de vue de l'intégration de la trajectoire ainsi que du calcul de Jacobien pour la méthode de tir. Le cas particulier des arcs singuliers pour les problèmes de contrôle optimal est également traité.

Mots-clés : Problème aux deux bouts, second membre discontinu, contrôle optimal , méthode de tir, arcs singuliers

Shoot package v1.1 - User Guide

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06/2008

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1 Introduction

This package of Fortran 90 routines implements several variants of indirect shooting methods for optimal control problems, with an embedded homotopy approach (discrete continuation). Special care is given to the numerical integration of the shooting function and the precise computation of its Jacobian, especially in the discontinuous case. Originally designed for the resolution of bang-bang low thrust orbital transfers for the CNES[3], the software has been since then extended to handle problems with singular arcs. The integration of state constraints is currently under development.

1.1 Shooting method

We begin with a brief presentation of the single shooting method, which is part of the *indirect* methods, and is based on Pontryagin's Maximum Principle (we refer readers interested in these methods to [17, 7, 2, 19] for instance). We recall that *direct* methods, on the other hand, typically involve the partial (control) or total (control and state) discretization of the problem, and then use various approaches (SQP and interior point techniques for instance) to solve the resulting optimization problem. Direct methods are thus supposed to be robust, but the counterpart is a relatively low precision, and a huge problem size depending on the discretization stepsize used. This makes these methods ill-suited to some particular cases, such as problems with a bang-bang control structure and a huge number of commutations.

Back to the indirect methods, single shooting consists in finding a zero of the *shooting function* associated with the original problem. There is no discretization, even if the method still involves an integration of the system in some way. It is a fast and high precision method, but often requires a good initial guess: as they typically consist in applying a (quasi-)Newton solver to the shooting function, the convergence radius may be quite small, depending on the problem. This is particularly true for problems involving commutations, or worse, singular arcs, and often leads to the use of continuation (homotopy) techniques to obtain a suitable initial point.

We consider a general optimal control problem in the Bolza form

$$(P)\begin{cases} Min\ g(t_0, x(t_0), t_f, x(t_f)) + \int_{t_0}^{t_f} l(t,x,u)\ dt & Objective \\ \dot{x} = f(t,x,u) & Dynamics \\ u \in U & Admissible\ Controls \\ \psi_0(t_0, x(t_0)) = 0 & Initial\ Conditions \\ \psi_1(t_f, x(t_f)) = 0 & Terminal\ Conditions \end{cases}$$

Notation/Remark: for clarity, the time t will often be omitted in the formulas, except in ambiguous cases.

We use here and in all the following the notations: $x \in \mathbf{R}^n$ for the state, $u \in \mathbf{R}^m$ for the control, U the set of admissible controls, and f for the state dynamics. We assume that the initial and final times t_0 and t_f are fixed. Free

[3]The French space agency

final time problems can always be reformulated as fixed time problems using the classical time transform $t = s.t_f$ with $s \in [0,1]$ the normalized time and t_f an additional state variable. We introduce the costate p, of same dimension as the state x, and define the **Hamiltonian** by

$$H(t,x,p,u) = l(t,x,u) + (p|f(t,x,u)).$$

Remark: we thus assume in all the following that we are in the ***normal*** *case, meaning that the costate p_0 associated to the integral objective l is non zero, and can be made equal to 1, by dividing the whole costate p by p_0...*

Pontryagin's Maximum Principle then states that, under the assumptions:
• $\exists\ (x,u)$ feasible for (P), with x absolutely continuous and u measurable.
• f and l are continuous with respect to u and C^1 with respect to t and x.
• g, ψ_0, ψ_1 are C^1 with respect to x.

Let $(\overline{x}, \overline{u})$ be an optimal pair for (P), then

(i) $\exists\ \overline{p} \neq 0$ absolutely continuous such that we have the Hamiltonian system

$$\begin{cases} \dot{\overline{x}} = \frac{\partial H}{\partial p}(t, \overline{x}, \overline{p}, \overline{u}) \\ \\ \dot{\overline{p}} = -\frac{\partial H}{\partial x}(t, \overline{x}, \overline{p}, \overline{u}) \end{cases}$$

(ii) $\overline{u}$ is solution of $Min_{w \in U} H(t, \overline{x}, \overline{p}, w)$ ae in $[t_0, t_f]$.

(iii) "Transversality conditions": $\exists\ (\overline{\mu}_0, \overline{\mu}_1)$ such that

$$(TC) \begin{cases} \psi_0(x(t_0)) = 0 \\ \overline{p}(t_0) = -\frac{\partial \Phi}{\partial x_0}(t_0, \overline{x}(t_0), t_f, \overline{x}(t_f), \overline{\mu}_0, \overline{\mu}_1) \\ \\ \psi_1(x(t_f)) = 0 \\ \overline{p}(t_f) = \frac{\partial \Phi}{\partial x_f}(t_0, \overline{x}(t_0), t_f, \overline{x}(t_f), \overline{\mu}_0, \overline{\mu}_1) \end{cases}$$

with

$$\Phi : (t_0, x_0, t_f, x_f, \mu_0, \mu_1) \mapsto g(t_0, x_0, t_f, x_f) + (\psi_0(t_0, x_0)|\mu_0) + (\psi_1(t_f, x_f)|\mu_1)$$

Remark: this is why indirect methods are sometimes referred to as necessary condition *methods.*

Now we denote $y = (x, p)$ and φ the state-costate dynamics derived from the Hamiltonian system. We assume here that the expression of the optimal control given by the necessary conditions (Hamiltonian minimization) is actually a function, noted γ, ie $\overline{u}(t) = ArgMin_{w \in U} H(t, x, p, w) = \gamma(t, y)$. Solving (P) is equivalent to solving the following Boundary Value Problem

$$(BVP) \begin{cases} \dot{y} = \varphi(t, y, \gamma(t, y)) & ae\ in\ [t_0, t_f] \\ c_0(t_0, y(t_0)) = 0 & Boundary\ Conditions\ at\ t_0 \\ c_1(t_f, y(t_f)) = 0 & Boundary\ Conditions\ at\ t_f \end{cases}$$

Note: these Boundary Conditions c_0 and c_1 correspond to the Transversality Conditions mentioned above, that contain the Initial and Terminal conditions

of (P) *in addition to the constraints on the costate* p*.*

It is possible to integrate $y = (x, p)$ if we set the value of $y(t_0)$, and we then obtain the following Initial Value Problem

$$(IVP) \begin{cases} \dot{y} = \varphi(t, y, \gamma(t, y)) & ae\ in\ [t_0, t_f] \\ y(t_0) = \zeta & Initial\ Value. \end{cases}$$

We introduce now an application called the **shooting function**, which basically maps the initial value ζ to the value of the Boundary conditions at t_f for the corresponding solution of (IVP). In practice, the initial conditions ψ_0 of the problem (P) already give a part of $y(t_0)$, so the unknown of the shooting function is reduced to the "missing" part, that we note z. A frequent situation is when the initial conditions determine the initial state $x(t_0)$, therefore z is actually the initial costate $p(t_0)$. Then the value of the shooting function is given by the boundary conditions at t_f for the solution $y(\cdot, z)$ of (IVP) corresponding to the initial value $y(t_0, z) = (x_0, z)$

$$(Shooting\ function) \qquad S : z \mapsto c_1(y(t_f, z)).$$

Finding a zero of the shooting function S is then equivalent to the resolution of (BVP), and therefore also gives a solution of (P). The "shooting method" thus consists in solving the equation $S(z) = 0$, as summarized below:

Problem (P)	$\longrightarrow$	Boundary Value Problem (BVP)	$\longrightarrow$	Initial Value Problem (IVP) and Shooting function S

An interesting particular case is when the optimal control is discontinuous and presents switchings (or commutations), for instance when the command law is bang-bang. More precisely, the Hamiltonian minimization gives

$$\begin{array}{lll} u = \gamma_1(t, y) & , & if\ \psi(t, y) < 0 \\ u = \gamma_2(t, y) & , & if\ \psi(t, y) > 0 \\ u = \Gamma(t, y) & , & if\ \psi(t, y) = 0, \end{array}$$

with $\Gamma(t, y)$ a subset of U, and ψ the switching function, whose zeros correspond to the commutations of the optimal control. We assume that the set of switching times t such that $\psi(t, y) = 0$ is finite. We then obtain $y(\cdot, z)$ as solution of the initial value problem with a discontinuous right hand side

$$(IVP)_{disc} \begin{cases} \dot{y} & = \varphi_1(t, y) & , & if\ \psi(t, y) < 0 \\ \dot{y} & = \varphi_2(t, y) & , & if\ \psi(t, y) > 0, \\ y(t_0, z) & = (x_0, z) & & \end{cases}$$

As mentioned earlier, one of the main practical difficulties for the shooting method is to find a suitable initial point, due to the small convergence radius of the newton method applied to the shooting function. In our case, we use a continuation approach (homotopy), as described below.

1.2 Continuation

It is well known that shooting methods are quite sensible to the initial guess, and it is often quite hard to achieve convergence for non trivial problems because of this. An interesting way to obtain a suitable starting point is to use

continuation (or *homotopy*) approaches. The principle of continuation is to find a certain homotopy that connects the original problem to an easier one, and then to follow the zero path of this homotopy, from a solution of the easy problem to a solution of the original one.

In the following, we will often parametrize the original problem (P) by a certain $\lambda \in [0,1]$, and obtain a problem family (P_λ) such that $(P_1) = (P)$. Moreover, we expect that we are able to find a solution of (P_0). If we note S_λ the shooting function associated to (P_λ), we can define the *homotopy*

$$\mathcal{H} : (z, \lambda) \mapsto S_\lambda(z).$$

Assuming we know a zero z_0 of $\mathcal{H}(\cdot, 0) = S_0$, all we have to do is to follow the zero path of $\mathcal{H}$ from $\lambda = 0$ to $\lambda = 1$, at which point we have obtained a zero of $\mathcal{H}(\cdot, 1) = S_1 = S$, and therefore a solution of (P).

There are several ways to perform this path following, such as Predictor-Corrector methods (or differential homotopy) that treat the zero path as a differentiable curve, Piecewise Linear (or simplicial) methods that build a PL approximation of the path over a triangulation of the space. We refer the interested readers to [1] for a general overview of continuation methods, and [8] in the context of Newton methods for nonlinear problems.

A more basic approach is just to try to solve a sequence of problems $(P_{\lambda_k}), k = 1, \ldots, N$ such that $\lambda_0 = 0$ and $\lambda_N = 1$, by using the solution found for (P_{λ_k}) as initial guess for $(P_{\lambda_{k+1}})$, or with a basic linear prediction step. The stepsize h_k for the homotopic parameter (such that $\lambda_{k+1} = \lambda_k + h_k$) can be set to a certain fixed value, or something a little more refined. For instance we can introduce a tolerance *tol* on the norm of the shooting function that indicates a successful convergence of the shooting method. Then we can start with a stepsize $h_0 = 1$ and try to solve (P_1) directly; if the attempt fails, we divide the stepsize h_k by 2, and try again, until we reach $\lambda = 1$. It is then safer to also stop the continuation if a certain maximal number of iterations has been reached, or if the stepsize h_k goes below some minimal value. This method is implemented in the Shoot package, and can be quite effective for well chosen homotopies.

2 Using the Shoot package

2.1 Package overview

Web page: **http://www.cmap.polytechnique.fr/~martinon/**
Contact: **pierre.martinon@inria.fr**, **joseph.gergaud@enseeiht.fr**
All questions or comments about the Shoot package are welcome !

The Shoot package should contain the following files:

Core files

- *ShootCont.f90*: discrete continuation module
- *Shoot.f90*: shooting functions module

- *Integrators.f90*: integrators interface
- *RHS.f90*: IVP right hand side handling
- *Common.f90*: some common subroutines
- *ShootDefs.f90*: global variables
- *Makefile*

Sample problem files

- *OrbitalFuns.f90*: orbital transfert with discontinuous control
- *FishingFuns.f90*: fishing problem with singular arcs
- *RegulatorFuns.f90*: quadratic regulator problem with singular arcs
- *GoddardFuns.f90*: 3D Goddard problem with singular arcs

Visualization scripts

- *sol.m*: Matlab script for solution visualization (use *solorb.m* for orbital transfer)
- *contpath.m*: Matlab script for continuation path visualization

Third party codes

- *rkf45.f*: Runge Kutta Fehlberg (4-5) integrator
  L. Shampine, H. Watts,
  http://netlib.org/ode/
- *dopri5.f*: Dormand Prince (5-4) integrator
  dop853.f: Dormand Prince (8-5-3) integrator
  odex.f: Gragg Bulirsch Stoer extrapolation integrator
  radau5.f: Radau (stiff) integrator
  E. Hairer, G. Wanner,
  http://elib.zib.de/pub/elib/hairer-wanner/
- *hybrd.f* and *hybrj.f*: Hybrid Powell method for solving nonlinear systems
  B. Garbow, K. Hillstrom, J. More,
  http://netlib.org/minpack/
- *tn.f*: Truncated Newton method for bounded optimization
  S. Nash,
  http://netlib.org/opt/
- *lbfgs.f*: Limited memory BFGS method for bounded optimization
  C. Zhu, R. Byrd, P. Lu-Chen, J. Nocedal,
  http://netlib.org/opt/
- *inverse.f*: matrix inversion (LU factorization, dgetri.f and dgetref.f)
  http://netlib.org/lapack/
- *auxsubs.f*: miscellaneous auxiliary subroutines

User guide

- this user guide

Compilation notes

Requirements: Fortran 95 compiler.
On Unix/Linux platforms, just use the *make* command to build the executables.

Note: the package has been tested with the following fortran compilers:
- ifort (Intel Fortran compiler) - gfortran (gcc.gnu.org/fortran/) - g95 (www.g95.org)

2.2 User supplied subroutines and input files

In order to solve an optimal control problem with the Shoot package, the user has to complete the following subroutines, located in *ProblemFuns.f90*, and described below:

- **InitPar**: performs specific problem initializations, if any.
- **Control**: optimal control and switching function evaluation.
- **Dynamics**: state, costate and objective dynamics.
- **Switch**, **SwitchDot**: optional, compute the switching function and its time derivative (needed for switching detection or singular arcs).

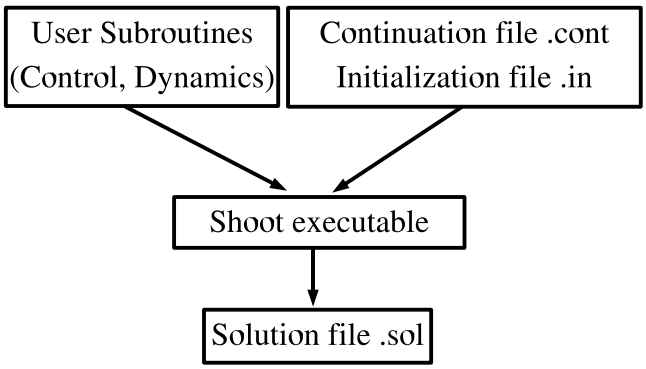
Then for each problem, two input files are used as well:

- **Problem initialization file .in**: problem data and shooting method settings.
- **Continuation file .cont**: discrete continuation settings.

After execution, the code will produce the output file:

- **Solution file .sol**: state, costate, control (and switching function) from the initial time to the final time, for the solution of the shooting method (can be visualized with the provided scripts).
- **Continuation Path file .contpath**: the sequence of solutions generated during the discrete continuation (can be visualized with the provided scripts).

Note: for each problem, a single prefix name is associated to all input and output files. Thus if the input file are named *Demo1.in* and *Demo1.cont* for instance, the corresponding output files will be *Demo1.sol* and *Demo1.contpath*. This makes it easier to keep track of which problem the files correspond to.



Once user-supplied subroutines and input files are completed, just run the executable. The program will ask the name (without the extension) of the input files, for instance "Problem1" for *Problem1.cont* and *Problem1.in*. Keep in mind that all input and output files associated to a given problem have the same name, only with different extensions. Thus it is quite easy to keep track of several problems at the same time, each one having its set of files with the same prefix.

At the end of the execution, a solution file **.sol** is generated, that contains the solution at the end of the continuation. The script *sol.m* traces the state, costate and control with respect to time, as well as the switching function optionally. A second output file **.contpath** contains the continuation path, that can be traced with the *contpath.m* script.

2.2.1 InitPar

This subroutine performs the various initializations required by the problem, and can be left empty. It is called at the beginning of the program, and then each time the shooting function is computed.

The input variables are
- *mode*: 0 for first call, 1 for subsequent calls
- z: shooting function unknown
- *lambda*: homotopic parameter used for continuation

Subroutine interface

```
Subroutine InitPar(mode, z, lambda)
  implicit none

  integer, intent(in) :: mode
  real(kind=8), intent(in) :: lambda
  real(kind=8), dimension(n), intent(in) :: z
```

2.2.2 Control

This subroutine provides the value of the optimal control, according to necessary conditions (Hamiltonian minimization)

The input variables are
- *lambda*: homotopic parameter used for continuation
- t: time
- x: state (dimension ns)
- p: costate (dimension nc)

And the output variables
- u: optimal control (dimension m)

The external parameter *controlmode* indicates what the subroutine should compute at each call: - 0: compute standard control
- 1: compute bang-bang control according to external parameter $switchflag = sign(\psi)$.

This mode is used only when switching detection is enabled. Typical values for *switchflag* are -1 and 1, and the control should be set accordingly. An incoming value of 0 indicates that *switchflag* must be initialized to match the sign of the switching function, in addition to computing the control.
- 2: compute the exact expression of the control over a singular arc. This mode is used only when one or more singular arcs are specified in the control structure (see 14) and the singular control mode is set to use the exact expression (see 15).

Subroutine interface

```
Subroutine Control(lambda,t,x,p,u)
  implicit none

  real(kind=8), intent(in) ::  lambda, t
  real(kind=8), intent(in), dimension(ns) :: x
  real(kind=8), intent(in), dimension(nc) :: p
  real(kind=8), intent(out), dimension(m) :: u
```

2.2.3 Dynamics

This subroutine provides the dynamics for the state, costate, and objective.

The input variables are
- *dimphi*: dynamics dimension (> ns+nc when objective is required)
- *lambda*: homotopic parameter used for continuation
- t: time
- x: state (dimension ns)
- p: costate (dimension nc)
- u: the optimal control (dimension m)

And the output variable
- *phi*: dynamics (state, then costate, and optionally objective)

Subroutine interface

```
Subroutine Dynamics(dimphi,lambda,t,x,p,u,phi)
  implicit none

  integer, intent(in) :: dimphi
  real(kind=8), intent(in) :: lambda, t
  real(kind=8), intent(in), dimension(ns) :: x
  real(kind=8), intent(in), dimension(nc) :: p
  real(kind=8), intent(in), dimension(m) :: u
  real(kind=8), intent(out), dimension(dimphi) :: phi
```

2.2.4 Switch

This subroutine provides the switching function that is required for the switching detection or the handling of singular arcs.
The input variables are
- *lambda*: homotopic parameter used for continuation
- t: time
- x: state (dimension ns)

- p: costate (dimension nc)

And the output variable
- *psi*: switching function

Subroutine interface

```
Subroutine Switch(lambda,x,p,psi)
  implicit none
  real(kind=8), intent(in) ::  lambda
  real(kind=8), intent(in), dimension(ns) :: x
  real(kind=8), intent(in), dimension(nc) :: p
  real(kind=8), intent(out) :: psi
```

For problems with singular arcs, the first time derivative of the switching function is also required. The subroutine *SwitchDot* must then be completed as well (same interface as *Switch*).

2.2.5 Continuation file .cont

- *Problem dimension*
  Integer: same as the Unknown dimension in the intialization file

- *Initial and final value for homotopic parameter*
  Real — **Real**: Initial value should be the same as λ_0 in the intialization file; the discrete continuation will try to reach the final value set here for λ.

- *Tolerance, max iterations and iterations output frequency*
  Real — **Integer** — **Integer**: tolerance indicates the norm threshold on the shooting function for a continuation step to be accepted. The two integers set the maximum number of iterations for the continuation, and the output frequency (a value of n means displaying info every n iteration.

- *Debug and verbose flags*
  Integer — **Integer** [**0..2**]: debug should be set to 0 for normal use; the verbose flag can be set to 1 or 2 to display some additional output.

2.2.6 Initialization file .in

This file contains problem initializations, such as dimensions, initial and terminal conditions, initial guess, and integrator choices. Here is a brief summary of the parameters set in this file.

- *Homotopy choice and problem class*
  Integer [**1+**]: tells which homotopy is to be used by the program.
  Set to 1 by default, this permits to use several different homotopies for a given problem with the same *ProblemFuns.f90* file.
  Integer [**1+**]: sets the kind of shooting method used.
  1: Single shooting
  2+: Multiple shooting (under development)

- *Unknown, State, Costate and Control dimensions*
  Integer — **Integer** — **Integer** — **Integer**: here are specified the dimensions of the shooting function unknown, and of the state, costate and control of the problem (the state and costate dimensions are usually the same, but can differ if some components are known to be constant, and therefore do not need to be integrated).

- *Objective and Switch dimension*
  Integer [1+]: usually set to 1, as the objective is scalar, but additional objectives can also be computed, if the corresponding derivatives are included in subroutine **Dynamics**.
  Integer [0+]: like the objective, the switching function ψ has scalar values, but its derivatives can also be computed, if set in subroutine **Control** (observing the switching function is especially useful when suspecting the presence of singular arcs). For instance, a value of 2 indicates that both ψ and $\dot{\psi}$ are available.

- *Number of IVP unknown values*
  Integer: dimension of IVP unknown.
  IVP unknown indices
  Integer(): indices of IVP unknown values in (x, p) vector.

- *Number of initial values*
  Integer: number of values known at t_0 (given by initial and transversality conditions).
  Initial values indices
  Integer(): indices of these values in $(x(t_0), p(t_0))$.
  Initial values
  Real(): initial values.

- *Number of terminal values*
  Integer: number of values known at t_f (given by terminal and transversality conditions).
  Terminal values indices
  Integer(): indices of these values in $(x(t_f), p(t_f))$.
  Terminal values
  Real(): terminal values.

- *Fixed times*
  Integer: number of fixed times (often 2: initial and final time)
  Real — **Real**: values of fixed times.

- *Times structure*
  Integer [2+]: number of switching times (see below).
  Integer() List giving the type of switching times (see sample problems for examples of structures):
  0: initial of final time
  2: singular arc entry (-2 for light formulation)
  3: singular arc exit (-3 for light formulation)
  Full formulations use interior nodes and matching conditions; light formulations use only the switching times.

- *Starting point (z0,lambda0) for zeropath*
  Real(n+1): initial guess z_0 for shooting function unknown, and initial value λ_0 for the homotopic parameter λ.

- *Scaling mode*
  Integer [0..1]: selects the scaling mode.
  0: no scaling
  1: initial point scaling (to $[0.1, 1]$)

- *Integrator choice (type, steps, abstol, reltol)*
  Integer [0..9] — Integer — Real — Real

  The first integer selects the method for the numerical integration of the initial value problem (IVP)
  0: Fixed step - Euler
  1: Fixed step - Midpoint
  2: Fixed step - Runge Kutta 2nd order
  3: Fixed step - Runge Kutta 3rd order
  4: Fixed step - Runge Kutta 4th order
  5: Variable step - Runge Kutta Fehlberg 4-5th order (RKF45)
  6: Variable step - Dormand Prince 8-5-3 (DOP853)
  7: Variable step - Gragg Bulirsch Stoer extrapolation method (ODEX)
  8: Variable step - Dormand Prince 5-4 (DOPRI5)
  9: Variable step - Radau 5 (stiff) (RADAU5)

  The second integer indicates the number of steps for fixed step integrators (ignored for variable step integrators). The two remaining reals set the absolute and relative error tolerances for variable step integrators (ignored for fixed step integrators).

- *Switching detection*
  Integer

  This setting enables or disables the switching detection (cf page 16). Setting this parameter to 0 will disable switching detection (recommended if the right hand side of the IVP is continuous). A value of 1 will test for switchings at the end of each integration steps, higher values mean additionnal tests within each step.

- *Jacobian mode*
  Integer [0..1]

  This sets the method used to compute the Jacobian of the shooting function (cf page 17).
  0: classical finite differences on the shooting function
  1: variational equations (with finite differences for the right hand side)

- *Singular conditions and control*
  Integer [1..2] — Integer [0..2]

  The first integer gives the type of entry/exit conditions for singular arcs:
  1: requires $\psi = 0$ at entry and exit times (ψ is the switching function)
  2: requires $\psi = \dot{\psi} = 0$ at entry time ($\dot{\psi}$ must then be provided)

The second integer sets the way to compute the singular control:
0: exact singular control is provided
1: uses alternate singular control minimizing ψ^2
2: uses alternate singular control minimizing $\psi^2 + \dot{\psi}^2$

- *Number of parameters*
  Integer [0+]: number of problem specific parameters.
  Parameters
  Real(): problem specific parameters, if any.

3 Discontinuities handling

3.1 Switchings detection

Switching detection can be enabled if commutations (control discontinuities, typically for a bang-bang problem) are expected, in order to improve the integration precision and speed. Experiments on an orbital transfer problem (briefly presented below, see [11] for a more detailed study) show that this can significantly improve the convergence and speed of the shooting method.

Letting the integrator deal with the discontinuities of the right hand side on its own is often out of the question with a fixed step integrator, and even with a good variable step method, the cost can be heavy in terms of cpu time and accuracy. Therefore, we try to detect the switchings during the integration of (IVP). The principle of the detection is described below, and assumes that the integration method used is able to provide a "dense output", ie a cheap approximation of y on each integration interval. This is typically done by some polynomial interpolation, see [14] for a thorough description of dense outputs available for many integration methods.

At integration time t, compute the switching function $\psi(t, y)$.

If the sign of ψ has changed since the previous step, a commutation has occurred (actually, an odd number of commutations, but most often one in practice).

Then we locate the commutation by dichotomy, by using the dense output of the integrator to provide y and ψ. Then we perform the control switch, store the sign of ψ, and restart the integration back from the commutation time.

Else the integration goes on normally.

This algorithm is quite simple, but effective. The main issue is the risk of skipping one or several commutations during an integration step. An easy improvement of the detection method is to perform additional sign checks at a fixed number of intermediate points, and not only at the end of the step. If

these additional checks bring some new commutations, then the number of intermediate checkpoints should be increased until no new commutations appear. Besides, with a variable step integrator, we can also expect the stepsize control mechanism to reject a step that would have missed commutations.

Notes:
- switching detection is activated by setting a value greater than 0 for the first parameter of the block *Switching detection and variational system* in the initialization file (this parameter is actually the number of sign checks for the switching function over each integration interval).
- the switching function ψ MUST be provided by the subroutine CONTROL.
- requires an integrator with dense output (currently RK3, RK4, DOPRI5 and RADAU5).

3.2 Variational System

The first way to compute the Jacobian of the shooting function is simply to use finite differences. We use for the shooting method the nonlinear solver HYBRD, which performs finite differences with a step of

$$h_j = \sqrt{\epsilon}|x_j|,$$

where ϵ is the error on the shooting function evaluation (for the numerical experiments, we set ϵ to the relative error of the integrator). This method is sometimes referred to as "external differentiation" (END). Its drawback is that when used with a variable step integrator, the integration steps can vary at the points where S is evaluated for the finite differences, which can impair the approximation of the Jacobian (see [4], [14] p.200).

Then, another possibility is to use the variational equations to compute the Jacobian of the shooting function. We first recall the smooth case, when we consider the derivative with respect to the initial condition of the system

$$(IVP) \begin{cases} \dot{y}(t) = \varphi(t, y(t)) \\ y_i(t_0) = y_i^0 \quad \text{for } i = 1, \dots, n \\ y_{i+n} = z_i \quad \text{for } i = 1, \dots, n. \end{cases}$$

If we note $y(\cdot, z)$ the solution of (IVP), we know that $\frac{\partial y}{\partial z_i}(t_f, z)$ is solution of the variational system

$$(VAR)_j \begin{cases} \dot{Y}_j(t) = \frac{\partial \varphi}{\partial y}(t, y(t)) Y_j(t) \\ Y_j(t_0) = \begin{pmatrix} 0 & \cdots & 1 & \cdots & 0 \end{pmatrix}^T \end{cases}$$

It is then possible to compute the Jacobian of the shooting function by integrating (VAR) along with (IVP). An interesting possibility is to approximate the right hand side by finite differences (cf [14] p.201):

$$\dot{Y}(:, j) \approx \frac{1}{h} \left(\varphi(y + hY(:, j)) - \varphi(y)\right)$$

The use of the variational equations can be adapted to the discontinuous case, see [11].

Notes:
- the use of variational equations instead of external differentiation is activated by setting the *Jacobian mode* parameter in the initialization file to 1.
- if switchings are expected, switching detection MUST be enabled !
- this option uses the HYBRJ solver instead of HYBRD.

3.3 Sample problem: Orbital transfer

3.3.1 Problem statement

We study here an orbital transfer problem, originally submitted by CNES[4]. We consider a satellite with a mass of 1500 kg and low thrust electro-ionic propulsion (with thrusts ranging from 10 Newtons to 0.1 Newton). We want to transfer it from a strongly elliptic, slightly inclined orbit, to a circular, equatorial, geostationary orbit. The objective is to maximize the payload, i.e., to minimize the fuel consumption during the transfer (this is not a minimum-time problem). This kind of problems was studied for instance in [3, 16], and this specific family of problems in [11, 9, 10].

Model

We consider that the forces applied to the satellite are the Earth attraction (central force) and the engine thrust:

$$\ddot{r} = -\mu \frac{r}{\|r\|^3} + \frac{T}{m}$$

with r the position vector in $\mathbf{R}^3$, T the thrust also in $\mathbf{R}^3$ (ie the control), m the mass of the satellite, μ the gravitational constant of the Earth.

State

In order to avoid the strong oscillations that would result from the high number of revolutions, we use orbital parameters instead of Cartesian coordinates to define the state variables:
• Orbit parameter P
• Eccentricity vector (e_x, e_y), in the orbit plane, oriented towards perigee
• Rotation vector (h_x, h_y), in the equatorial plane, collinear to the intersection of orbit and equatorial planes
• True longitude $L = \Omega_n + \omega + w$
• Mass m

Control

The normalized three-dimensional control u such that $T = T_{max}\ u$ is expressed in the moving reference frame attached to the satellite, as radial thrust q, transverse thrust s and normal thrust w.

Optimal Control Problem

[4]Centre National d'Etudes Spatiales

If we note $x = (P, e_x, e_y, h_x, h_y, L)$, with T_{max} the maximal thrust we have the following formulation of the problem:

$$(P) \begin{cases} Min \int_{t_0}^{t_f} \|u(t)\| \ dt & \\ \dot{x} = f_0(x) + \frac{T_{max}}{m} B(x)u & \dot{m} = -\beta T_{max} \ \|u\| \\ \|u\| \leq 1 & \\ x(t_0) = (11625, 0.75, 0, 0.0612, 0, \pi) & m(t_0) = 1500 \\ x(t_f) = (42165, 0, 0, 0, 0, free) & m(t_f) \ is \ free \\ t_0 = 0 & \\ t_f \ is \ fixed & \end{cases}$$

with f_0 and B depending on x, and $\beta = 0.05112 km^{-1} s$.

Switching function and Control

We note p the costate associated to x, and p_m the costate associated to m, and define the **switching function** ψ:

$$\psi(x, m, p, p_m) = 1 - \beta T_{max} p_m \ - \ \frac{T_{max}}{m} \|B(x)^t p\|.$$

The application of Pontryagin's Maximum Principle then leads to the following expression of the optimal control

$$\begin{cases} u = -\frac{B(x)^t p}{\|B(x)^t p\|} & if \ \psi(x, m, p, p_m) < 0 \\ u = 0 & if \ \psi(x, m, p, p_m) > 0 \end{cases}$$

We can see that this control has a bang-bang structure, as its norm switches between 0 and 1 at zeros of the switching function ψ. The two cases define the two dynamics φ_1 and φ_2.

3.3.2 Continuation

We choose here for the "easy" problem the same transfer with minimization of an "energy" criterion, namely $\int_{t_0}^{t_f} \|u(t)\|^2 \ dt$. For our homotopy, we take the shooting function corresponding to the orbital transfer with the following objective, parametrized by $\lambda \in [0, 1]$:

$$J_\lambda = \int_{t_0}^{t_f} \ \lambda \|u(t)\| + (1 - \lambda)\|u(t)\|^2 \ dt.$$

The resulting perturbed problem (P_λ) has a strictly convex Hamiltonian (with respect to u), with a continuous optimal control, and is much easier to solve than $(P) = (P_1)$. As we shall see below, the shooting method indeed converges for (P_0) from a simple initial guess. Moreover, with the switching detection enabled, the solution for (P_0) is a sufficient starting point for (P), so the continuation is here reduced to its simplest form.

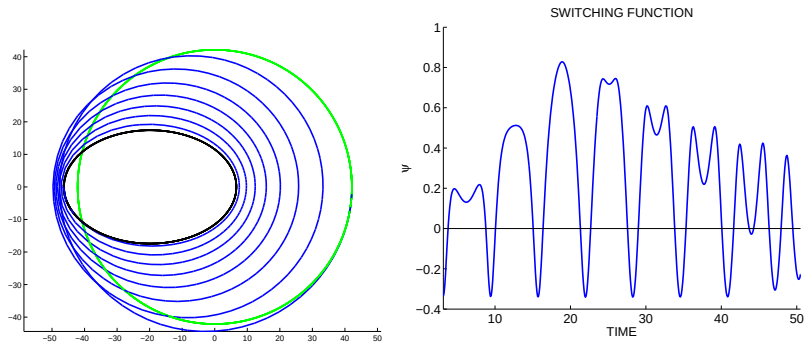
3.3.3 Numerical results

We present here briefly the numerical results for the orbital problem, for a maximal thrust of $10N$ and $0.1N$. For both problems we see that the shooting method converges for $\lambda = 0$ from the given starting point, as indicated in the "Initial Shoot" output block. Then the algorithm proceeds to the discrete continuation, and in this case directly converges for $\lambda = 1$ with the solution from $\lambda = 0$, therefore only 1 iteration is needed.

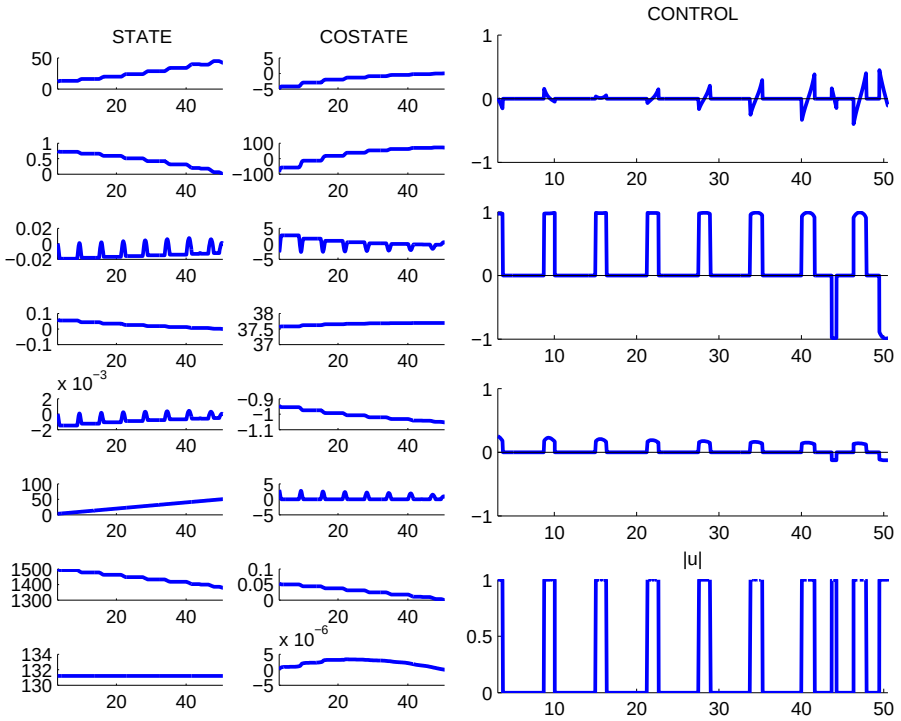
Note: the executable for the orbital transfer problem is named "Orb", and the input files for these two problems are given in Appendix, see page 28

```
 Orb
 Continuation file Problem.cont not found, enter problem files prefix:
transfer-10N

 Testing Initial Shooting point 3.4124398979055597

 Initial Shoot...
 HYBRD solver returns:  3 NFEV: 63 (NJAC: 3 )
 Solution:
 153.86708075216225
 -7.726744873083644
 -211.85656730659204
 -2.5120549807886023
 34.94893110107479
 -0.11932701650713314
 4.038568508459959
 0.060170453641508456
 0.
 Shooting function norm at the solution:  1.616224597310514E-15

 Starting discrete homotopy...
 HYBRD solver returns:  1 NFEV: 76 (NJAC: 5 )
 Iteration:    1    lambda: 1.000000E+00    Norm: 4.767508E-14
 Discrete Homotopy successfull...
 Shooting function norm at the solution:  4.76750773437339E-14

 Iterations:  1  Execution time:  5.714132
 Generating solution...
 Homotopy norm at lambda= 1. : 4.76750773437339E-14
 Switchings detected... 18 0 0 0 0 0 0 0 0 0
 Criterion value: 121.21183107816664
```

We show the solution (state, costate and control) obtained for $\lambda = 1$, where the control switchings (18) are clearly visible on the control and switching function graphs. We also draw the trajectory of the satellite around the earth (about 7 revolutions)



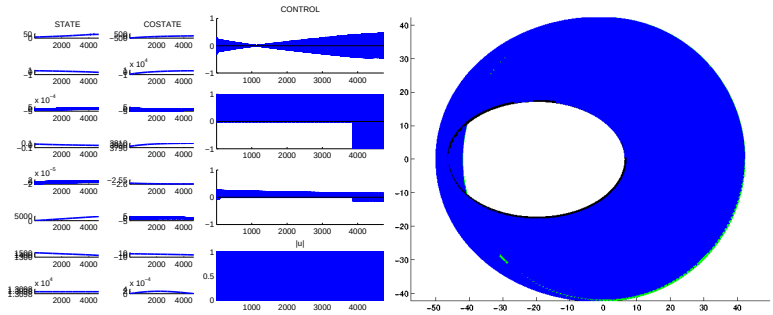
Orbital transfer (10N) - Trajectory and switching function



Orbital transfer (10N) - State, costate and control

We study now the same problem, but with a maximal thrust of only $0.1N$. Notice that the transfer time, number of revolutions (i.e. final longitude) and number of switchings are inversely proportional to the thrust, therefore we find values a hundred times greater than for the 10N transfer. Also, the value of the criterion (minimal fuel consumption) remains the same as for the 10N transfer.

We draw once again the solution and trajectory, which have the same structure than for the 10N transfer, but with 1800 switchings, 750 revolutions around the earth, and a final time of 640 days (!).



Orbital transfer (0.1N) - State, costate and control, and trajectory

Here is the execution log for the 0.1N transfer (input files are also given in appendix). We notice that the cpu time is also roughly a hundred times higher than for the 10N transfer, which is mainly due to the fact that the integration of (IVP) to compute the value of the shooting fonction is a hundred times longer. For this family of orbital transfer problems, experiments with maximal thrust ranging from $10N$ to $0.1N$ show that the number of Jacobian evaluations needed for the shooting method (both at $\lambda = 0$ and $\lambda = 1$) is roughly the same for all thrusts, between 3 and 6. Moreover, the discrete continuation always converges directly after 1 iteration.

```
S3D
 Continuation file Problem.cont not found, enter problem files prefix:
transfer-01N

 Testing Initial Shooting point 3.180517515056758

 Initial Shoot...
 HYBRD solver returns:  3 NFEV: 86 (NJAC: 5 )
 Solution:
 15315.789639285988
 -782.6478275916782
 -21386.64012621194
 -3.97023354622747
 3564.663731700098
 -1.229486020055539
 4.1759348571570065
 6.083187841162088
 0.
 Shooting function norm at the solution:  2.0559776184205325E-14

 Starting discrete homotopy...
 HYBRD solver returns:  1 NFEV: 62 (NJAC: 4 )
 Iteration:    1    lambda: 1.000000E+00    Norm: 7.394970E-12
 Discrete Homotopy successfull...
 Shooting function norm at the solution:  7.394970348098505E-12

 Iterations:  1  Execution time:  548.85956
 Generating solution...
 Homotopy norm at lambda= 1. : 7.394970348098505E-12
 Switchings detected... 1814 0 0 1 0 0 0 0 0 0
 Criterion value: 121.70246011040842
```

4 Singular arcs handling

4.1 Singular arcs and continuation

Among optimal control problems, singular arcs problems are interesting and difficult to solve with indirect methods, as they involve a multi-valued control and differential inclusions. In short, singular arcs occur in indirect methods when the necessary conditions from the Hamiltonian minimization fails to determine uniquely the optimal control over a non-trivial interval. In this case, applying the Pontryagin's Maximum Principle leads to a Boundary Value Problem with a differential inclusion.

Fortunately, there exist some well-known additional conditions that allow us to determine the value of the singular control, see [13, 18, 6]. In particular, exploiting the fact that the successive time derivatives of $\frac{\partial H}{\partial u}$ must vanish over a singular arc usually give the expression of the singular control (this is in practice often done by derivating the switching function).

The shooting method can be adapted to solve this kind of problems, but this typically requires some a priori assumptions on the control structure (number and approximate location of singular arcs in particular), and a good starting point. So we look for a method able to provide a reliable singular structure detection and initialization.

We resort once again to a continuation approach, and regularize the problem by adding a quadratic (u^2) perturbation to the criterion. We limit here ourselves to the case where the Hamiltonian is affine with respect to the control (which is quite common when singular arcs are involved), so that the regularized Hamiltonian is strongly convex with respect to the control. This ensures that the optimal control u^* is continuous, without any commutations or singular arcs, which makes the regularized problem much easier to solve.

The aim here is not to perform the continuation until the regularization vanishes, which would be impossible anyway due to the presence of singular arcs in the original problem. We just want to obtain sufficient information concerning the control structure, and an acceptable starting point for the shooting method.

This approach is illustrated below on a 3D version of the famous Goddard problem.

4.2 Shooting problem formulation

The shooting problem formulation has to be adapted to take singular arcs into account. The structure of the control must be prescribed by assigning a fixed number of interior junction times between nonsingular and singular arcs. These times $(t_i)_{i=1..n_{switch}}$ are part of the unknowns and must satisfy some junction conditions. Optionaly, the value of the state and costate at the start of each interior arc may also be added to the shooting unknowns. In this case matching conditions must be verified at the junction times (typically state and costate

continuity).

A junction condition indicates a change of structure, which corresponds here to an extremity of a singular arc. Along a singular arc, it is required that the switching function ψ and its time derivatives vanish. In practice, the singular control is often computed using the relation $\ddot{\psi} = 0$, therefore switching conditions may consist for instance in requiring either $\psi = 0$ at both extremities of the singular arc, or $\psi = \dot{\psi} = 0$ at the beginning of the arc (tangential entry). In our simulations, we choose the latter solution which happens to provide better and more stable results.

Additional unknowns:

- interior junction times $(t_i)_{i=1,nswitch}$

optiona state and costate values at interior times

Additional equations:

- switching conditions at junction times

optional matching conditions at interior times

Summary: shooting function unknown and value layout (here with the starting points for each interior arc)

Unknown z	IVP unknown at t_0	(x^1, p^1)	(x^2, p^2)	...	Δ_1	Δ_2	...

- usual IVP unknown at t_0
- values of (x^i, p^i) at interior junction times t_i
- interior junction times $(t_i)_{i=1,nswitch}$

Value $S_{Struct}(z)$	$Junction_{cond}(t_1)$	$Match_{cond}(t_1)$	...	Conditions at t_f

- junction and matching conditions at interior times
- usual terminal and transversality conditions at t_f

4.3 Sample problem: Goddard (3D)

4.3.1 Problem statement

Note: the full study of the following problem is described in [5].

For the third example, we consider an extension of the classical Goddard's problem (maximizing the final altitude of a rocket with vertical trajectory, the controls being the norm and direction of the thrust force, see [12, 15]) for non-vertical trajectories. We want here to maximize the final mass (i.e., minimizing the fuel consumption) while steering the system from a given initial point to a fixed final position, the final velocity vector and final time being free.

$$
\begin{array}{l}
Min \int_0^{t_f} \|u(t)\| dt \\
\dot{r} = v, \\
\dot{v} = -\frac{D(r,v)}{m} \frac{v}{\|v\|} - g(r) + C\frac{u}{m}, \\
\dot{m} = -b\|u\| \\
\|u(t)\| \leq 1 \\
r(0) = r_0,\ v(0) = v_0,\ m(0) = m_0 \\
r(t_f) = r_f,\ v(t_f)\ free, m(t_f)\ free \\
t_f\ free
\end{array}
$$

where the state variables are $r(t) \in \mathbf{R}^3$ (position of the spacecraft), $v(t) \in \mathbf{R}^3$ (velocity vector) and $m(t)$ (mass of the engine). Also, $D(r, v) > 0$ is the drag component, $g(r) \in \mathbf{R}^3$ is the usual gravity force, and $b > 0$ is a positive real number depending on the engine. The thrust force is $Cu(t)$, where $C > 0$ is the maximal thrust, and the control is the normalized thrust in $\mathbf{R}^3$.

Depending on the features of the problem (initial and final conditions, mass / thrust ratio, etc), it is known that control strategies that consist in choosing the control so that $\|u(t)\|$ is piecewise constant all along the flight, either equal to 0 or to the maximal authorized value 1, may not be optimal, as a consequence of the high values of the drag for high speed. Optimal trajectories may indeed involve singular arcs, on which the thrust is neither null nor maximal.

The Hamiltonian of the optimal control problem is

$$H = \langle p_r, v \rangle + \left\langle p_v, -\frac{D(r,v)}{m}\frac{v}{\|v\|} - g(r) + C\frac{u}{m} \right\rangle - (1 + bp_m)\|u\|,$$

which leads to the switching function $\psi(t) = \frac{C}{m(t)}\|p_v(t)\| - (1 + bp_m(t))$,

- if $\psi(t) < 0$ then $u(t) = 0$;
- if $\psi(t) > 0$ then $u(t) = \frac{p_v(t)}{\|p_v(t)\|}$;
- if $\psi(t) = 0$ on $I \subset [0, t_f]$, then the control u is singular, and

$$u(t) = \alpha(t)\frac{p_v(t)}{\|p_v(t)\|} \quad \text{a.e. on } I$$

Furthermore, on a singular subarc, derivating the switching function twice yields the expression of α via a relation of the form $A(x, p)\alpha = B(x, p)$. The computations are actually quite tedious to do by hand, and we used the symbolic calculus tool MAPLE. The expressions of A and B are quite complicated and are not reported here.

The free final time problem is formulated as a fixed final time one via the usual time transformation $t = t_f\ s$, with $s \in [0, 1]$ and t_f an additional component of the state vector, such that $\dot{t_f} = 0$ and $t_f(0), t_f(1)$ are free, with the associated costate satisfying $\dot{p}_{t_f} = -H$. All the graphs in the following will use this normalized time interval $[0, 1]$.

Transversality conditions on the adjoint vector yield $p_v(1) = (0\ 0\ 0)$, $p_m(1) = 0$, and $p_{t_f}(0) = p_{t_f}(1) = 0$. The unknown of the shooting function S is therefore

$$z = (t_f, p_r(0), p_v(0), p_m(0)) \in \mathbf{R}^8 \quad .$$

As usual, we regularize the cost function by considering an homotopic connection with an energy,

$$\int_0^{t_f} \left(\|u(t)\| + (1-\lambda)\|u(t)\|^2\right) dt,$$

where the parameter of the homotopy is $\lambda \in [0, 1]$. The resulting perturbed problem (P_λ) has a strongly convex Hamiltonian (with respect to u) for $\lambda < 1$, with a continuous optimal control.

4.3.2 Numerical results

In this case, even solving the regularized problem (P_0) is not obvious, due to the aerodynamic forces (drag). For this reason, we introduce a preliminary continuation on the atmosphere density, starting from a problem without atmosphere. This first homotopy can be initialized with a very simple intial guess (see Appendix for the input files) and converges without any difficulties to a solution of the regularized problem.

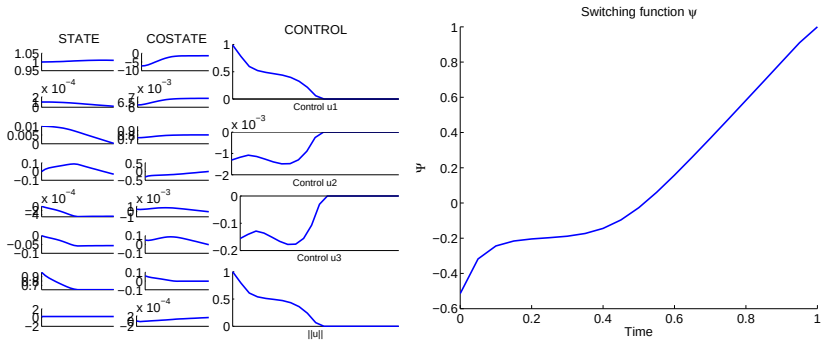
```
Sgod
 Continuation file Problem.cont not found, enter problem files prefix:
g3d-atm

 Testing Initial Shooting point 0.22015101337776394

 Initial Shoot...
 Solution:
 0.2196601318339915
 -2.1982603830073453
 0.005992036550913937
 0.599203655091394
 -0.4680695437269313
 0.0013395929123425805
 0.13395929123425804
 0.09461339086969081
 0.
 Shooting function norm at the solution:  3.3073150759414176E-17

 Starting discrete homotopy...
 Iteration:    1    lambda: 1.000000E+00    Norm: 1.080271E-02
 Iteration:    2    lambda: 5.000000E-01    Norm: 1.278126E-14
 Iteration:    3    lambda: 1.000000E+00    Norm: 7.620364E+02
 Iteration:    4    lambda: 7.500000E-01    Norm: 7.502007E+01
 Iteration:    5    lambda: 6.250000E-01    Norm: 5.209608E-14
 Iteration:    6    lambda: 7.500000E-01    Norm: 5.704429E-14
 Iteration:    7    lambda: 8.750000E-01    Norm: 6.660700E-13
 Iteration:    8    lambda: 1.000000E+00    Norm: 2.223085E-13
 Discrete Homotopy successfull...

 Final Shoot...
 Solution:
 0.2640799154711017
 -8.12234874767145
 0.007775791055176457
```

```
0.7775791055163178
-0.4779458411988672
0.0005715193702808507
0.05715193702811029
0.0995842151720355
1.
Shooting function norm at the solution:  2.2230846942024136E-13

Iterations:  8  Execution time:  22.645557
```

We can now perform the main continuation on the quadratic regularization, using the solution obtained after the atmosphere homotopy as initial guess. We choose to stop the continuation after 250 iterations, which corresponds here roughly to $\lambda \approx 0.8$.

```
 Sgod
 Continuation file Problem.cont not found, enter problem files prefix:
g3d-reg

 Testing Initial Shooting point 0.02515351520206591

 Initial Shoot...
 Solution:
0.2640793552246857
-8.123438706780812
0.0077766675936303715
0.7776667593630372
-0.47805884594351794
0.0005716753713626369
0.05716753713626369
0.09959301100188714
0.
Shooting function norm at the solution:  1.2282561281514605E-13

 Starting discrete homotopy...
 Iteration:   50    lambda: 4.726562E-01    Norm: 1.964334E-04
 Iteration:  100    lambda: 6.015625E-01    Norm: 7.694480E-13
 Iteration:  150    lambda: 6.992188E-01    Norm: 2.323932E-03
 Iteration:  200    lambda: 7.724609E-01    Norm: 4.539889E-04
 Iteration:  250    lambda: 7.998047E-01    Norm: 3.834804E-03
 Maximum number of iterations reached... 250

 Final Shoot...
 Solution:
0.22752675243050716
-7.259631372627164
0.006215831510268079
0.7375636170012954
-0.3125538050315717
0.00041468499277015207
0.04930418817757816
0.058110013804430705
0.7998046875
Shooting function norm at the solution:  0.002854535614040327

Iterations:  250  Execution time:  148.066491
```

Here is the solution at the end of the continuation. The control and switching function graphs strongly suggest the presence of a singular subarc around the time interval $[0.1, 0.4]$.

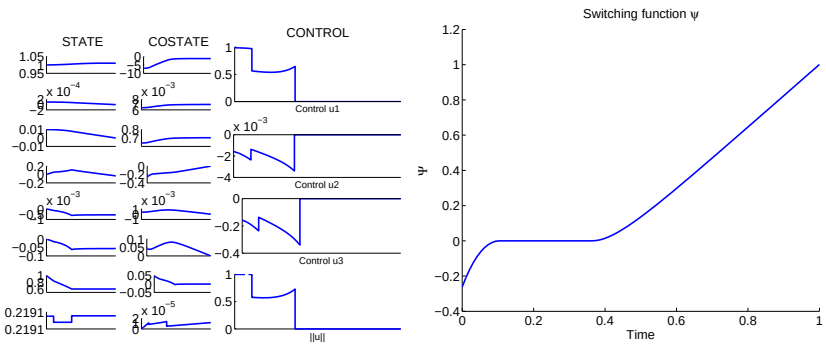


Solution at the end of the continuation for $\lambda \approx 0.8$

Using the information from the previous solution at $\lambda = 0.8$, we assume a control structure of the form *bang-singular-bang* and try the singular shooting, that converges to a solution with the expected singular arc.

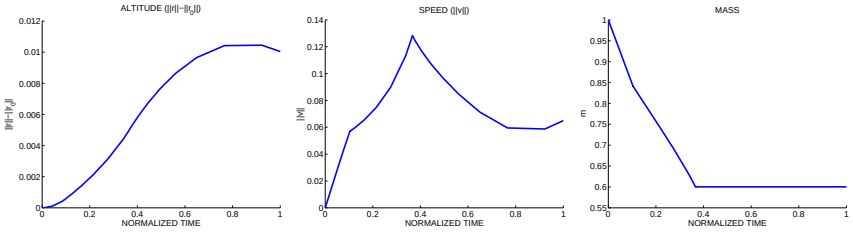
```
Sgod
Continuation file Problem.cont not found, enter problem files prefix:
g3d-shoot

Testing Initial Shooting point 3.8612091309731476

Initial Shoot...
HYBRD solver returns:  5 NFEV: 499 (NJAC: 8 )
Solution:
// (solution removed for more clarity ...) //
1.
Shooting function norm at the solution:  0.0002645393058478628

Iterations:  0  Execution time:  3.0875310000000002
Generating solution...
switchtime*tf 0. 0.022774715513691285 0.07977732948936873 0.2191378890189921
Switching condition at switch  1 at 0.10392869811626898
is -0.00000190814054401577795 -0.00002932641104428758
Homotopy norm at lambda= 1. : 0.0002645393058478628
Singular shooting objective...
Criterion value: 0.4002680426684089
```

We show the state, costate, control and switching function for the solution with a singular arc.



Goddard (3D) - State, costate, control and switching function

The altitude, speed and mass during the flight are represented below.



Solution with singular arc: altitude, speed and mass

5 Appendix: Files for the sample problems

5.1 Orbital transfer

Energy to mass homotopy - transfer-10N.cont

```
Problem dimension
8
Initial and final value for homotopic parameter
0.0 1.0
Tolerance, max iterations and iterations output frequency
1d-3 1 1
Debug and verbose flags
0 1
```

Energy to mass homotopy - transfer-10N.in

```
Homotopy choice and problem class
1 1
Unknown, State, Costate and Control dimensions
8 8 8 3
Objective and Switch dimension
1 1

Number of IVP unknown values
8
IVP unknown indices
8 9 10 11 12 13 14 15

Number of initial values
8
Initial values indices
1 2 3 4 5 6 7 16
Initial values
11.625 0.75 0 0.0612 0 0 1500 0

Number of terminal values
8
Terminal values indices
1 2 3 4 5 6 15 16
Terminal Values
42.165 0 0 0 0 1 0 0

Fixed times
2
3.14159 29.89539

Times structure
2
0 0

Starting point (x0,lambda0) for zeropath
150.0
0.1 0.1 0.1 0.1 0.1 0.1 0.1
0.0

Scaling mode
1
Integrator choice (type, steps, abstol, rtol)
4 500 1d-8 1d-8

Switching detection
10
Jacobian mode
0
Singular conditions and control
0 0

Number of parameters
7
Parameters
10 1.77 0.0142 5165.86 12.96 1 0
```

For the 0.1N transfer, the continuation file is the same as for the 10N transfer. The only differences for the initialization file are the final "time" (actually longitude), the initial guess for t_f (first component of z_0), the number of integration steps for the RK4 method, and of course the maximal thrust (first of the parameters at the end of the file).

Energy to mass homotopy - transfer-01N.in

```
Homotopy choice and problem class
1 1
Unknown, State, Costate and Control dimensions
8 8 8 3
Objective and Switch dimension
1 1

Number of IVP unknown values
8
IVP unknown indices
8 9 10 11 12 13 14 15

Number of initial values
8
Initial values indices
1 2 3 4 5 6 7 16
Initial values
11.625 0.75 0 0.0612 0 0 1500 0

Number of terminal values
8
Terminal values indices
1 2 3 4 5 6 15 16
Terminal Values
42.165 0 0 0 0 1 0 0

Fixed times
2
3.14159 2678.52159

Times structure
2
0 0

Starting point (x0,lambda0) for zeropath
15000.0
0.1
0.1
0.1
0.1
0.1
0.1
0.1
0.0

Scaling mode
1
Integrator choice (type, steps, abstol, rtol)
4 50000 1d-8 1d-8

Switching detection
10
Jacobian mode
0
Singular conditions and control
0 0

Number of parameters
7
Parameters
0.1 1.77 0.0142 5165.86 12.96 1 0
```

5.2 Goddard problem

Atmosphere homotopy - g3d-atm.cont

```
Problem dimension
8
Initial and final value for homotopic parameter
0.0 1.0
Tolerance, max iterations and iterations output frequency
1d-6 666 1
Debug and verbose flags
0 0
```

Atmosphere homotopy - g3d-atm.in

```
Homotopy choice and problem class
2 1
Unknown, State, Costate and Control dimensions
8 8 8 3
Objective and Switch dimension
1 1

Number of IVP unknown values
8
IVP unknown indices
8  9 10 11  12 13 14  15

Number of initial values
8
Initial values indices
1 2 3   4 5 6   7  16
Initial values
0.999949994d0 1d-4 1d-2 0.d0 0d0 0d0 1d0  0d0

Number of terminal values
8
Terminal values indices
1 2 3   12 13 14  15 16
Terminal Values
1.01d0 0d0 0d0  0d0 0d0 0d0  0d0 0d0

Fixed times
2
0d0 1d0

Times structure
2
0 0

Starting point (x0,lambda0) for zeropath
1d-1
1d-1 1d-1 1d-1 1d-1 1d-1 1d-1 1d-1
0.0

Scaling mode
1
Integrator choice (type, steps, abstol, rtol)
4 100 1d-6 1d-4

Switching detection
0
Jacobian mode
0
Singular conditions and control
1 0

Number of parameters
5
Parameters
3.5d0 2d0 310 1 0
```

Regularization homotopy - g3d-reg.cont

```
Problem dimension
8
Initial and final value for homotopic parameter
0.0 1.0
Tolerance, max iterations and iterations output frequency
1d0 250 50
Debug and verbose flags
0 0
```

Regularization homotopy - g3d-reg.in

```
Homotopy choice
1 1
Unknown, State, Costate and Control dimensions
8 8 8 3
Objective and Switch dimension
1 2

Number of IVP unknown values
8
IVP unknown indices
8  9 10 11  12 13 14 15

Number of initial values
8
Initial values indices
1 2 3 4 5 6 7  16
Initial values
0.999949994d0 1d-4 1d-2 0d0 0d0 0d0 1d0  0d0

Number of terminal values
8
Terminal values indices
1 2 3  12 13 14   15 16
Terminal Values
1.01d0 0d0 0d0  0d0 0d0 0d0  0d0 0d0

Fixed times
2
0d0 1d0

Times structure
2
0 0

Starting point (x0,lambda0) for zeropath
 0.2640799154711017
-8.12234874767145    0.007775791055176457   0.7775791055163178
-0.4779458411988672  0.0005715193702808507  0.05715193702811029
 0.0995842151720355
0

Scaling mode
1
Integrator choice (type, steps, abstol, rtol)
4 100 1d-6 1d-6

Switching detection
0
Jacobian mode
0
Singular conditions and control
2 0

Number of parameters
5
Parameters
3.5d0 2d0 310 1 0
```

Singular shooting - g3d-arc.cont

```
Problem dimension
42
Initial and final value for homotopic parameter
1. 1.
Tolerance, max iterations and iterations output frequency
1d-3 66 1
Debug and verbose flags
0 1
```

Singular shooting - g3d-arc.in

```
Homotopy choice and problem class
1 1
Unknown, State, Costate and Control dimensions
42 8 8 3
Objective and Switch dimension
1 2

Number of IVP unknown values
8
IVP unknown indices
8 9 10 11 12 13 14 15

Number of initial values
8
Initial values indices
1 2 3 4 5 6 7 16
Initial values
0.999949994 1d-4 1d-2 0 0 0 1 0

Number of terminal values
8
Terminal values indices
1 2 3  12 13 14   15 16
Terminal Values
1.01d0 0d0 0d0  0d0 0d0 0d0   0d0 0d0

Fixed times
2
0d0 1d0

Times structure
4
0 2 3 0

Starting point (x0,lambda0) for zeropath
  0.21840416845860117
  -6.9877785293508952
  6.74841991325446453E-003
  0.67337518271960106
 -0.26819091932383449
  4.38522398496401567E-004
  4.38868519728624965E-002
  4.97853303547632908E-002

 1.00053391896293E+00
 9.84996726170234E-05
 9.84979144653172E-03
 4.95108744565758E-02
-1.36710643692133E-04
-1.36898965856133E-02
 8.53768851611924E-01
 2.18404168458601E-01
-6.30734390402251E+00
 6.82561959408012E-03
 6.81096347528290E-01
-1.94365698301661E-01
 4.62462663477968E-04
 4.63357092451593E-02
 3.21748625699720E-02
 6.15293199902717E-05
```

```
 1.00341668398768E+00
 8.74106767276750E-05
 8.73850429782001E-03
 8.40431373992978E-02
-4.08427882061274E-04
-4.09554713842058E-02
 6.76620375538197E-01
 2.18404168458601E-01
-3.07896616808134E+00
 7.15670640150622E-03
 7.14210048315405E-01
-1.67712312265266E-01
 7.74192385038949E-04
 7.77242943243487E-02
 1.13331798546918E-02
 1.92008073724007E-04

0.1
0.3
 1.00000000000000E+00

Scaling mode
1
Integrator choice (type, steps, abstol, rtol)
4 100 1d-6 1d-6

Switching detection
0
Jacobian mode
0
Singular conditions and control
2 0

Number of parameters
5
Parameters
3.5d0 2d0 310 0 1
```

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